

# SPHERICAL MEANS IN ODD DIMENSIONS AND EPD EQUATIONS

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ABSTRACT. The paper contains a simple proof of the Finch-Patch-Rakesh inversion formula for the spherical mean Radon transform in odd dimensions. This transform arises in thermoacoustic tomography. Applications are given to the Cauchy problem for the Euler-Poisson-Darboux equation with initial data on the cylindrical surface. The argument relies on the idea of analytic continuation and known properties of Erdélyi-Kober fractional integrals.

## 1. INTRODUCTION

We consider the spherical mean Radon transform

$$(1.1) \quad (Mf)(\theta, t) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(\theta - t\sigma) d\sigma,$$

$$\theta \in S^{n-1}, \quad t \in \mathbb{R}_+ = (0, \infty),$$

where  $f$  is a smooth function supported inside the unit ball  $B = \{x \in \mathbb{R}^n : |x| < 1\}$ ,  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$  with the area  $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ , and  $d\sigma$  denotes integration against the usual Lebesgue measure on  $S^{n-1}$ . The inversion problem for this transform has attracted considerable attention in the last decade in view of new developments in thermoacoustic tomography; see [AKK, AKQ, FPR, FHR, KK, Ku, PS1, PS2] and references therein. Explicit inversion formulas for  $Mf$  in the “closed form” are of particular interest. For  $n$  odd, such formulas were obtained by Finch, Patch, and Rakesh in [FPR]. The corresponding formulas for  $n$  even were obtained by Finch, Haltmeier, and Rakesh in [FHR]. An interesting inversion formula of different type, that covers both odd and even cases, was suggested by

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Kunyansky [Ku]; see also a survey paper [KK], where diverse inversion algorithms and related mathematical problems are discussed.

In spite of their elegance and ingenuity, most of explicit inversion formulas for  $Mf$  are still mysterious, and the basic ideas behind them are not completely understood. We observe, for example, that the derivation for  $n = 3$  in [FPR] relies on implementation of delta functions and is not completely rigorous. The dimensions  $n = 5, 7, \dots$  have been treated using a pretty complicated reduction to the 3-dimensional case. The resulting inversion formula can be written in the form

$$(1.2) \quad f(x) = c_n \Delta \int_{S^{n-1}} [D^{n-3} t^{n-2} \varphi_\theta] \Big|_{t=|x-\theta|} d\theta,$$

where  $\varphi_\theta(t) = (Mf)(\theta, t)$ ,

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}, \quad c_n = \frac{(-1)^{(n-1)/2}}{4\pi^{n/2-1} \Gamma(n/2)}, \quad D = \frac{1}{2t} \frac{d}{dt};$$

cf. [FPR, Theorem 3]. Other formulas in that theorem can be obtained in the framework of the same method; see Remark 3.1.

In the present article (Section 3) we suggest a simple and rigorous proof of (1.2) that handles all odd  $n$  simultaneously, so that no reduction to  $n = 3$  is needed. The idea is to treat the spherical mean as a member of a certain analytic family of operators associated to the Euler-Poisson-Darboux equation

$$(1.3) \quad \square_\alpha u \equiv \Delta_x u - u_{tt} - \frac{n + 2\alpha - 1}{t} u_t = 0$$

and invoke known facts from fractional calculus [SKM] about Erdélyi-Kober operators. Section 4 deals with applications. Here we reconstruct the solution  $u(x, t)$  of the equation  $\square_\alpha u = \lambda^2 u$ ,  $\lambda \geq 0$ , from the Cauchy data on the cylinder  $S^{n-1} \times \mathbb{R}_+$ . Section 5 contains comments and open questions.

No claim about the originality of the results presented in this article is made, but it is felt that the elementary use of operators of fractional integration to obtain them might appeal to the applied mathematician. If the reader is not interested in applications in Section 4, he may skip subsections 2.2 and 2.4. These are not needed for the basic Section 3.

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## 2. PRELIMINARIES

**2.1. Erdélyi-Kober fractional integrals.** We remind some known facts; see, e.g., [SKM, Sec. 18.1]. For  $\operatorname{Re} \alpha > 0$  and  $\eta \geq -1/2$ , the Erdélyi-Kober fractional integral of a function  $\varphi$  on  $\mathbb{R}_+$  is defined by

$$(2.1) \quad (I_\eta^\alpha \varphi)(t) = \frac{2t^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^t (t^2 - r^2)^{\alpha-1} r^{2\eta+1} \varphi(r) dr, \quad t > 0.$$

For our further needs, it suffices to assume that  $\varphi$  is infinitely smooth and supported away from the origin. Then  $I_\eta^\alpha \varphi$  extends as an entire function of  $\alpha$  and  $\eta$ , so that

$$(2.2) \quad I_\eta^0 \varphi = \varphi,$$

$$(2.3) \quad (I_\eta^\alpha \varphi)(t) = t^{-2(\alpha+\eta)} D^m t^{2(\alpha+m+\eta)} (I_\eta^{\alpha+m} \varphi)(t), \quad D = \frac{1}{2t} \frac{d}{dt},$$

$$(2.4) \quad (I_\eta^{-m} \varphi)(t) = t^{-2(\eta-m)} D^m t^{2\eta} \varphi(t),$$

where  $m$  is a nonnegative integer. The property

$$(2.5) \quad D^m = t^{-1} \left( \frac{d}{dt} \frac{1}{2t} \right)^m t$$

allows us to write (2.3) and (2.4) in a different equivalent form. The composition formula and the inverse operator are as follows:

$$(2.6) \quad I_{\eta+\alpha}^\beta I_\eta^\alpha \varphi = I_\eta^{\alpha+\beta} \varphi, \quad (I_\eta^\alpha)^{-1} \varphi = I_{\eta+\alpha}^{-\alpha} \varphi.$$

**2.2. The generalized Erdélyi-Kober fractional integrals.** Let  $J_\nu$  and  $I_\nu$  be the Bessel function and the modified Bessel function of the first kind, respectively [E]. The generalized Erdélyi-Kober operators are defined by

$$(2.7) \quad J_{\eta,\lambda}^\alpha \varphi(t) = t^{-2(\alpha+\eta)} J_\lambda^\alpha t^{2\eta} \varphi(t), \quad I_{\eta,\lambda}^\alpha \varphi(t) = t^{-2(\alpha+\eta)} I_\lambda^\alpha t^{2\eta} \varphi(t),$$

where  $\lambda \geq 0$ ,

$$(2.8) \quad J_\lambda^\alpha \varphi(t) = 2^\alpha \lambda^{1-\alpha} \int_0^t (t^2 - r^2)^{(\alpha-1)/2} J_{\alpha-1}(\lambda \sqrt{t^2 - r^2}) \varphi(r) r dr,$$

$$(2.9) \quad I_\lambda^\alpha \varphi(t) = 2^\alpha \lambda^{1-\alpha} \int_0^t (t^2 - r^2)^{(\alpha-1)/2} I_{\alpha-1}(\lambda \sqrt{t^2 - r^2}) \varphi(r) r dr;$$

see [L1, L2], [SKM, Sec. 37.2]. As above, we assume  $\varphi$  to be infinitely smooth and supported away from the origin. Integrals (2.8) and (2.9)

are absolutely convergent if  $\operatorname{Re} \alpha > 0$  and admit analytic continuation to all complex  $\alpha$  by the formulas

$$(2.10) \quad J_\lambda^\alpha \varphi = D^m J_\lambda^{\alpha+m} \varphi = J^{\alpha+m} D^m \varphi,$$

$$(2.11) \quad I_\lambda^\alpha \varphi = D^m I_\lambda^{\alpha+m} \varphi = I_\lambda^{\alpha+m} D^m \varphi,$$

$m \in \mathbb{N}$ . These follow from the well-known relation

$$\left( \frac{1}{\tau} \frac{d}{d\tau} \right)^m [\tau^\nu J_\nu(\tau)] = \tau^{\nu-m} J_{\nu-m}(\tau)$$

(similarly for  $I_\nu$ ). Clearly,  $J_{\eta,0}^\alpha = I_{\eta,0}^\alpha = I_\eta^\alpha$ . If  $\operatorname{Re} \alpha > 0$  and  $\operatorname{Re} \beta > 0$ , then, changing the order of integration and using the formula 2.15.15(1) from [PBM], we get

$$(2.12) \quad I_{\eta+\alpha,\lambda}^\beta J_{\eta,\lambda}^\alpha \varphi = I_\eta^{\alpha+\beta} \varphi.$$

The latter extends by analyticity to all complex  $\alpha$  and  $\beta$  and yields the inversion formula

$$(2.13) \quad (J_{\eta,\lambda}^\alpha)^{-1} f = I_{\eta+\alpha,\lambda}^{-\alpha} f.$$

By (2.11) and (2.7), this can also be written as

$$(2.14) \quad (J_{\eta,\lambda}^\alpha)^{-1} f = t^{-2\eta} D^m I_\lambda^{m-\alpha} t^{2(\eta+\alpha)} f = t^{-2\eta} D^m t^{2(\eta+m)} I_{\eta+\alpha,\lambda}^{m-\alpha} f.$$

**2.3. The Euler-Poisson-Darboux equation.** Consider the Cauchy problem for the Euler-Poisson-Darboux equation (1.3):

$$(2.15) \quad \square_\alpha u = 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

where  $f$  belongs to the Schwartz space  $\mathcal{S}(\mathbb{R}^n)$ ; see [B1] for details. If  $\alpha \geq (1-n)/2$ , then (2.15) has a unique solution  $u(x, t) = (M_t^\alpha f)(x)$  where the operator  $M_t^\alpha$  is defined in the Fourier terms by

$$[M_t^\alpha f]^\wedge(y) = m_\alpha(t|y|) \hat{f}(y),$$

$$m_\alpha(\rho) = \Gamma(\alpha + n/2) (\rho/2)^{1-\alpha-n/2} J_{n/2+\alpha-1}(\rho).$$

The operator  $M_t^\alpha$  extends meromorphically to all complex  $\alpha$  with the poles  $-n/2, -n/2 - 1, \dots$ . For  $\operatorname{Re} \alpha > 0$ , it is an integral operator

$$(2.16) \quad (M_t^\alpha f)(x) = \frac{\Gamma(\alpha + n/2)}{\pi^{n/2} \Gamma(\alpha)} \int_{|y| < 1} (1 - |y|^2)^{\alpha-1} f(x - ty) dy.$$

In the case  $\alpha = 0$ ,  $M_t^\alpha f$  is the spherical mean

$$(2.17) \quad (M_t^0 f)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f(x - t\sigma) d\sigma;$$

cf. (1.1). Passing to polar coordinates, one can obviously represent  $M_t^\alpha f$  as an Erdélyi-Kober integral of the spherical mean

$$(2.18) \quad (M_t^\alpha f)(x) = \frac{\Gamma(\alpha + n/2)}{\Gamma(n/2)} (I_\eta^\alpha \varphi_x)(t), \quad \varphi_x(t) = (M_t^0 f)(x),$$

with  $\eta = n/2 - 1$ .

**2.4. The generalized Euler-Poisson-Darboux equation.** Consider the more general Cauchy problem

$$(2.19) \quad \square_\alpha u = \lambda^2 u, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

where  $f$  is a Schwartz function and  $\lambda \geq 0$ . If  $\alpha \geq (1-n)/2$ , then (2.19) has a unique solution  $u(x, t) = (M_{t,\lambda}^\alpha f)(x)$ , where the operator  $M_{t,\lambda}^\alpha$  is defined as analytic continuation of the integral

$$(2.20) \quad (M_{t,\lambda}^\alpha f)(x) = \frac{(2/\lambda)^{\alpha-1} \Gamma(\alpha + n/2)}{\pi^{n/2}} t^{2-n-2\alpha} \\ \times \int_{|y|<t} f(x-y) (t^2 - |y|^2)^{(\alpha-1)/2} J_{\alpha-1}(\lambda \sqrt{t^2 - |y|^2}) dy;$$

see [B2] for details. As above,

$$(2.21) \quad (M_{t,\lambda}^\alpha f)(x) = \frac{\Gamma(\alpha + n/2)}{\Gamma(n/2)} (J_{\eta,\lambda}^\alpha \varphi_x)(t), \quad \varphi_x(t) = (M_t^0 f)(x),$$

$\eta = n/2 - 1$ , where  $J_{\eta,\lambda}^\alpha$  is the generalized Erdélyi-Kober operator (2.7).

**2.5. More preparations.** We restrict  $M_t^\alpha f$  to  $x = \theta \in S^{n-1}$  and set

$$(2.22) \quad (N^\alpha f)(\theta, t) = t^{n+2\alpha-2} (M_t^\alpha f)(\theta).$$

In particular, for  $\operatorname{Re} \alpha > 0$ , owing to (2.16), we have

$$(2.23) \quad (N^\alpha f)(\theta, t) = \frac{\Gamma(\alpha + n/2)}{\pi^{n/2} \Gamma(\alpha)} \int_{\mathbb{R}^n} f(y) (t^2 - |y - \theta|^2)_+^{\alpha-1} dy$$

where  $(\dots)_+^{\alpha-1}$  has a standard meaning, namely,  $(a-b)_+^{\alpha-1} = (a-b)^{\alpha-1}$  if  $a > b$  and 0 otherwise. Given a function  $F$  on the cylinder  $S^{n-1} \times \mathbb{R}_+$ , we denote

$$(2.24) \quad (PF)(x) = \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} F(\theta, |x - \theta|) d\theta, \quad x \in \mathbb{R}^n,$$

which is a modification of the back-projection (or dual) operator; cf. [H, N], where these notions are used for the classical Radon transform. We also invoke Riemann-Liouville integrals [SKM]

$$(2.25) \quad (I_{-1}^\alpha u)(s) = \frac{1}{\Gamma(\alpha)} \int_{-1}^s (s-t)^{\alpha-1} u(t) dt, \quad u \in C^\infty[-1, 1].$$

The integral (2.25) is absolutely convergent when  $\operatorname{Re} \alpha > 0$  and extends by analyticity to all  $\alpha \in \mathbb{C}$ , so that

$$(2.26) \quad (I_{-1}^{-m}u)(s) = (d/ds)^m u(s), \quad m = 0, 1, 2, \dots$$

**Lemma 2.1.** *Let  $B = \{x \in \mathbb{R}^n : |x| < 1\}$ . For  $\operatorname{Re} \alpha > 0$  and any integrable function  $f$  supported in  $B$  we have*

$$(2.27) \quad (PN^\alpha f)(x) = c_\alpha \int_B \frac{f(y)}{|x-y|^{1-\alpha}} (I_{-1}^\alpha u)(h) dy, \quad x \in B,$$

where

$$c_\alpha = \frac{\Gamma(n/2) \Gamma(\alpha + n/2)}{2^{1-\alpha} \pi^{(n+1)/2} \Gamma((n-1)/2)}, \quad h = \frac{|x|^2 - |y|^2}{2|x-y|}, \quad u(t) = (1-t^2)^{(n-3)/2}.$$

*Proof.* By (2.23) and (2.24), changing the order of integration, we get

$$\begin{aligned} (PN^\alpha f)(x) &= \frac{\Gamma(\alpha + n/2)}{\sigma_{n-1} \pi^{n/2} \Gamma(\alpha)} \int_{S^{n-1}} d\theta \int_B f(y) (|x-\theta|^2 - |y-\theta|^2)_+^{\alpha-1} dy \\ &= \frac{\Gamma(\alpha + n/2)}{\sigma_{n-1} \pi^{n/2} \Gamma(\alpha)} \int_B f(y) k_\alpha(x, y) dy, \end{aligned}$$

where

$$\begin{aligned} k_\alpha(x, y) &= \int_{S^{n-1}} (|x|^2 - |y|^2 - 2\theta \cdot (x-y))_+^{\alpha-1} d\theta \\ &= \frac{\sigma_{n-2}}{(2|x-y|)^{1-\alpha}} \int_{-1}^h (h-t)^{\alpha-1} (1-t^2)^{(n-3)/2} dt. \end{aligned}$$

This gives the result.  $\square$

### 3. INVERSION OF THE SPHERICAL MEAN FOR $n$ ODD

Let  $C^\infty(B)$  be the space of  $C^\infty$ -functions on  $\mathbb{R}^n$  supported in  $B$ ;  $f \in C^\infty(B)$ . By (2.22), (2.18), and (2.4), analytic continuation of  $N^\alpha f$  at  $\alpha = 3 - n$  has the form

$$(N^{3-n}f)(\theta, t) = \frac{\Gamma(3-n/2)}{\Gamma(n/2)} D^{n-3} t^{n-2} \varphi_\theta(t), \quad \varphi_\theta(t) = (Mf)(\theta, t).$$

If  $n = 2k + 3$ ,  $k = 0, 1, \dots$ , then (2.26) yields

$$(I_{-1}^{3-n}u)(h) = (d/dh)^{2k} (1-h^2)^k = (-1)^k k! = (-1)^{(n-3)/2} \Gamma(n-2).$$

Hence, analytic continuation of (2.27) at  $\alpha = 3 - n$  is

$$(3.1) \quad P[D^{n-3}t^{n-2}\varphi_\theta](x) = c(I^2f)(x), \quad c = 2(-1)^{(n-3)/2} \Gamma^2(n/2)/\pi,$$

where  $x \in B$  and

$$(3.2) \quad (I^2f)(x) = \frac{\Gamma(n/2 - 1)}{4\pi^{n/2}} \int_B \frac{f(y) dy}{|x-y|^{n-2}}$$

is the Riesz potential of order 2 [SKM]. The latter can be inverted by the Laplacian, and simple calculations yield

$$(3.3) \quad f(x) = c_n \Delta \int_{S^{n-1}} [D^{n-3} t^{n-2} \varphi_\theta] \Big|_{t=|x-\theta|} d\theta, \quad c_n = \frac{(-1)^{(n-1)/2}}{4\pi^{n/2-1} \Gamma(n/2)}.$$

*Remark 3.1.* Formula (3.3) coincides (up to notation) with the third formula in [FPR, Theorem 3]. Other formulas in that theorem can be similarly obtained from (2.27) if the latter is applied to  $\Delta f$  instead of  $f$ . Here we take into account that  $I^2 \Delta f = -f$ , because  $\text{supp} f$  is separated from the boundary of  $B$ .

#### 4. AN INVERSE PROBLEM FOR THE EPD EQUATION

Let  $\alpha \geq (1-n)/2$ ,  $\lambda \geq 0$ . Suppose we know the trace  $u(\theta, t)$  of the solution of the Cauchy problem

$$(4.1) \quad \square_\alpha u = \lambda^2 u, \quad u(x, 0) = f(x), \quad u_t(x, 0) = 0,$$

on the cylinder  $\{(\theta, t) : \theta \in S^{n-1}, t \in \mathbb{R}_+\}$  and want to reconstruct the initial function  $f$  in the space  $C^\infty(B)$ . This can be easily done using (3.1) and the Erdélyi-Kober operators. Indeed, by (2.21),

$$u(\theta, t) = (M_{t,\lambda}^\alpha f)(\theta) = \frac{\Gamma(\alpha+n/2)}{\Gamma(n/2)} (J_{\eta,\lambda}^\alpha \varphi_\theta)(t),$$

where  $\varphi_\theta(t) = (M_t^0 f)(\theta)$ ,  $\eta = n/2 - 1$ . Then by (2.13), for  $u_\theta(t) \equiv u(\theta, t)$  we have

$$\varphi_\theta = \frac{\Gamma(n/2)}{\Gamma(\alpha+n/2)} (J_{\eta,\lambda}^\alpha)^{-1} u_\theta = \frac{\Gamma(n/2)}{\Gamma(\alpha+n/2)} I_{\eta+\alpha,\lambda}^{-\alpha} u_\theta.$$

Now, (3.1) yields

$$\frac{\Gamma(n/2)}{\Gamma(\alpha+n/2)} P[D_t^{n-3} t^{n-2} I_{\eta+\alpha,\lambda}^{-\alpha} u_\theta] = c I^2 f, \quad c = 2(-1)^{(n-3)/2} \Gamma^2(n/2)/\pi,$$

and therefore,

$$(4.2) \quad f = \frac{(-1)^{(n-1)/2} \pi}{2\Gamma(n/2) \Gamma(\alpha+n/2)} \Delta P[D_t^{n-3} t^{n-2} I_{\eta+\alpha,\lambda}^{-\alpha} u_\theta].$$

The operator  $I_{\eta+\alpha,\lambda}^{-\alpha}$  can be replaced by any expression from (2.14).

It remains to note that once  $f$  is known, the solution  $u(x, t)$  of the equation  $\square_\alpha u = \lambda^2 u$  can be reconstructed from the trace  $u(\theta, t)$  by the formula  $u(x, t) = (M_{t,\lambda}^\alpha f)(x)$ ; see (2.20).

## 5. COMMENTS

1. It is a challenging open problem to appropriately adjust our method to the case when  $n$  is even and give alternative proof of the corresponding inversion formulas from [FHR] and [Ku].

2. Formula (3.1) provokes the following

**Conjecture.** *Let  $n \geq 3$  be odd. A function  $\varphi_\theta(t) \equiv \varphi(\theta, t)$  belongs to the range of the operator  $f \rightarrow (Mf)(\theta, t)$ ,  $f \in C^\infty(B)$ , if and only if  $P[D_t^{n-3}t^{n-2}\varphi_\theta]$  belongs to the range  $I^2[C^\infty(B)]$  of the potential (3.2).*

The “only if” part follows immediately from (3.1). The “if” part requires studying injectivity of the back-projection operator  $P$ , which is of independent interest; cf. [R], where injectivity and inversion of the dual Radon transform is studied in the general context of affine Grassmann manifolds. Various descriptions of the range of the spherical mean transform and many related results can be found in [AKQ, AK, FR].

3. It is worth noting that for  $n = 3$ , the inversion formula for  $Mf$  becomes elementary if  $f$  is a radial function, i.e.  $f(x) \equiv f_0(|x|)$ .

**Lemma 5.1.** *If  $f \in L^1_{loc}(\mathbb{R}^n)$ ,  $f(x) \equiv f_0(|x|)$ , then  $(Mf)(\theta, t) \equiv F_0(t)$ , where*

$$(5.1) \quad F_0(t) = \frac{2^{n-3} \Gamma(n/2)}{\pi^{1/2} \Gamma((n-1)/2)} \int_{|1-t|}^{1+t} f_0(r) [a(r, t)]^{n-3} r \, dr$$

$a(r, t) = [r^2 - (1-t)^2]^{1/2} [(1+t)^2 - r^2]^{1/2} / 4$  being the area of the triangle with sides  $1, t, r$ .

*Proof.*

$$\begin{aligned} (Mf)(\theta, t) &= \frac{1}{\sigma_{n-1}} \int_{S^{n-1}} f_0(|\theta - t\sigma|) \, d\sigma \\ &= \frac{\sigma_{n-2}}{\sigma_{n-1}} \int_{-1}^1 f_0(\sqrt{1+t^2-2ts}) (1-s^2)^{(n-3)/2} \, ds \end{aligned}$$

and (5.1) follows. □

**Corollary 5.2.** *If  $n = 3$  and  $f$  is supported in  $B$ , then (5.1) yields*

$$(5.2) \quad (a) \quad F_0(t) = \frac{1}{2t} \int_{1-t}^1 f_0(r) r \, dr \quad \text{if } 0 < t \leq 1,$$

and

$$(5.3) \quad (b) \quad F_0(t) = \frac{1}{2t} \int_{t-1}^1 f_0(r) r \, dr \quad \text{if } 1 \leq t < 2.$$



In the case (a),

$$(5.4) \quad f_0(r) = \frac{2}{r} \left[ \frac{d}{dt}(tF_0(t)) \right]_{t=1-r}.$$

In the case (b),

$$(5.5) \quad f_0(r) = -\frac{2}{r} \left[ \frac{d}{dt}(tF_0(t)) \right]_{t=1+r}.$$

*Remark 5.3.* We note that in (5.4) and (5.5) it is not necessary to know  $F_0(t)$  for all  $t \in (0, 2)$  as in (3.3). It suffices to know it only for  $t \in (0, 1)$  or  $t \in (1, 2)$ . It would be interesting to obtain similar inversion formulas for  $(Mf)(\theta, t)$  in the general case, when  $(Mf)(\theta, t)$  is known only for  $(\theta, t) \in S^{n-1} \times (0, 1)$  or  $(\theta, t) \in S^{n-1} \times (1, 2)$ , as in the radial case.

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